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α-Dedekind complete archimedean vector lattices versus α-quasi-F spaces

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Abstract

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α denotes an uncountable cardinal number, and $\mathcal{W}(\alpha)$ denotes the category of archimedean vector lattices with distinguished waek unit and unit preserving vector lattice homomorphisms. In this paper we show that in $\mathcal{W}(\alpha)$, the full subcategory of α-Dedekind complete objects is epireflective. We explain how the Yosida functor connects the algebraic notions of α-Dedekind complete, α-dense, and α-jam-dense with the topological notions of α-quasi-F, α-irreducible, and α-SpFi morphism. We then go on to show that the α-quasi-F cover of a compact space X, denoted $(\mathrm{QF}_\alpha X, q_\alpha)$, is the Yosida space of the α-Dedekind complete epireflection of $C(X)$ in $\mathcal{W}(\alpha)$. Finally we show that in the topological category α-SpFi, the full subcategory of α-quasi-F spaces is monocoreflective, and for each compact space X, $(\mathrm{QF}_\alpha X, q_\alpha)$ is the α-quasi-F monocoreflection of X.

Keywords: α-Dedekind complete vector lattice, α-quasi-F spaces, Yosida space, spaces with filters, α-jam-dense.

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1. Introduction

α denotes an uncountable cardinal number or the symbol ∞. The meaning of $\alpha=\infty$ will be clear from the context. We write $\alpha<\infty$ to mean that α is an arbitrary cardinal number.

$\mathcal{W}$ denotes the category of archimedean vector lattices with distinguished weak unit and unit preserving vector lattice homomorphisms. An element $u \in L \in |\mathcal{W}|$ is called a *weak unit* if the band (complete ideal) generated by u is all of L [32, 10]. u is called a *strong unit* if the principal ideal generated by u is all of L. $\mathcal{S}$ denotes the category of archimedean vector lattices with distinguished strong unit and unit preserving vector lattice homomorphisms. Obviously, a strong unit is a weak unit, and $\mathcal{S}$ is a subcategory of $\mathcal{W}$. L^+ denotes all the *strictly* positive elements of L, i.e., $L^+ = \{a \in L: a > 0\}$.

An archetypical $\mathcal{W}$ object is $C(X)$ (the ring of continuous real-valued functions on a topological space X [11]) with the constant function $\mathbf{1}$ taken as the distinguished weak unit. Note that the weak unit $\mathbf{1}$ is indeed a strong unit, and $(C(X), \mathbf{1})$ is an $\mathcal{S}$ object.

Definition 1.1. A $\mathcal{W}$ morphism $\varphi: L \to M$ is called *α-complete* if, for $A \subset L$ with $|A| < \alpha$, we have that $\varphi(\bigvee^L A) = \bigvee^M \varphi[A]$ whenever $\bigvee^L A$ exists in L. $\mathcal{W}(\alpha)$ denotes the category of $\mathcal{W}$ objects with α-complete morphisms.

Definition 1.2. An $L \in |\mathcal{W}|$ is called *α-Dedekind complete* (also α-jamd-complete [5]) if whenever $A, B \subset L$, $|A|, |B| < \alpha$, $B \geqslant A$, and $0 = \bigwedge (B - A)$, then there is a $c \in L$ such that $c = \bigvee A = \bigwedge B$. (We write $B \geqslant A$ to mean $b \geqslant a$ for all $a \in A$, $b \in B$, and $B - A$ for $\{b - a: b \in B, a \in A\}$.)

Recall L is called *Dedekind complete* if every subset of L that is bounded above has a supremum in L. L is ∞-Dedekind complete (as defined above) if and only if L is Dedekind complete. Two primary results of this paper are:

Theorem 2.16. *Let $\gamma: L \hookrightarrow M$ be an α-jam-dense (see Proposition 2.5) embedding. Suppose $\varphi: L \to N$ is α-complete and N is α-Dedekind complete. Then there exists a unique α-complete morphism $\bar{\varphi}: M \to N$ such that $\varphi = \bar{\varphi} \circ \gamma$.*

Theorem 2.19. *In $\mathcal{W}(\alpha)$, the full subcategory of α-Dedekind complete objects is epireflective.*

Other significant results, as we explain below, are the "topological realizations" of the two results above.

Definition 1.3. Comp denotes the category of compact Hausdorff spaces and continuous functions. In [2, 3, 15, 16], the contravarient *Yosida functor*, $Y: \mathcal{W} \to \text{Comp}$ is investigated. For each $L \in |\mathcal{W}|$, there is a $Y(L) \in |\text{Comp}|$, called the Yosida space of L [32]; and for each $\mathcal{W}$ morphism $\varphi: L \to M$ there is a continuous function $Y(\varphi): Y(M) \to Y(L)$.

Definition 1.4. Let $X, Y \in |\text{Comp}|$. A continuous function $f: X \to Y$ is called an α-SpFi *morphism* if $f^{-1}(K)$ is dense in X whenever K is dense and α-Lindelöf in Y. α-SpFi [4, 5, 20] denotes the topological category of compact Hausdorff spaces with α-SpFi morphisms.

Below is a result which is essentially [3, 4.2]. It shows that Y restricted to $\mathcal{W}(\alpha)$ is a functor to α-SpFi.

Theorem 3.10. *A $\mathcal{W}$ morphism $\varphi : L \to M$ is α-complete if and only if $Y(\varphi) : Y(M) \to Y(L)$ is an α-*SpFi *morphism.*

Moreover, Y takes $\mathcal{W}(\alpha)$ epics to α-SpFi monics:

Theorem 3.11. *An α-complete $\mathcal{W}$ morphism $\varphi : L \to M$ is epic in $\mathcal{W}(\alpha)$ if and only if $Y(\varphi) : Y(M) \to Y(L)$ is monic in α-*SpFi.

Definition 1.5. A Tynchonoff topological space X is called *α-quasi-F* if each dense α-Lindelöf set in X is C^*-embedded (ω_1-quasi-$F \equiv$ quasi-F [9, 18]). X is ∞-quasi-F if and only if X is extremally disconnected [11, 28].

Theorem 3.13. *Let $L \in |\mathcal{W}|$. If L is α-Dedekind complete, then $Y(L)$ is α-quasi-F.*

The converse of this statement is not true; however:

Theorem 3.12 [5]. *X is α-quasi-F if and only if $C(X)$ is α-Dedekind complete.*

Applying Y to Theorem 2.19, we obtain the topological result:

Theorem 4.3. *In α-*SpFi, *the full subcategory of α-quasi-F spaces is monocoreflective.*

Definition 1.6. A *cover* of a compact space X is a pair (Y, f), where Y is a compact space, and f maps Y irreducibly onto X. A continuous surjection f is *irreducible* if f does not map any proper closed subset of the domain onto the codomain. For an abstract topological property P, we say (Y, f) is a *P cover* of X if (Y, f) is a cover of X, and Y has the property P. (Y, f) is called the minimum P cover of X if given any other P cover (Z, g) of X, there is a unique irreducible map $h : Z \to Y$ such that $g = f \circ h$.

In this paper (Theorem 4.3), we identify the α-quasi-F cover of a compact space X, denoted $(\mathrm{QF}_\alpha X, q_\alpha)$, as the Yosida space of the α-Dedekind complete epireflection, in $\mathcal{W}(\alpha)$, of $C(X)$. Moreover, in α-SpFi, $(\mathrm{QF}_\alpha X, q_\alpha)$ is the α-quasi-F monocoreflection of X.

2. The α-Dedekind complete epireflection

In this section, Zorn's lemma is used to obtain the α-Dedekind complete epireflection of L. L, M, and N denote $\mathcal{W}$ objects, and maps between them are considered to be $\mathcal{W}$ morphisms unless otherwise stated. The following is straightforward.

Proposition 2.1. *Let $\varphi : L \to M$. The following are equivalent.*

(a) *φ is α-complete.*

(b) *There is a $c \in L$ such that whenever $B \subset L$, $|B| < \alpha$, and $c = \bigvee^L B$, then $\varphi(c) = \bigvee^M \varphi[B]$.*

(c) *For $A \subset L$ with $|A| < \alpha$, we have that $\varphi(\bigwedge^L A) = \bigwedge^M \varphi[A]$ whenever $\bigwedge^L A$ exists in L.*

Proposition 2.2. *Let $\varphi : L \to M$ and $\gamma : M \hookrightarrow N$. If $\gamma \circ \varphi$ is α-complete, then φ is α-complete.*

Proof. Let $c \in L$ and suppose $c = \bigvee^L A$ where $A \subset L$ and $|A| < \alpha$. We claim that $\varphi(c) = \bigvee^M \varphi[A]$. Suppose not. Then there is a $b \in M$ such that $\varphi(c) > b > \varphi(a)$ for all $a \in A$. Because γ is injective, we have $\gamma \circ \varphi(c) > \gamma(b) > \gamma \circ \varphi(a)$ for all $a \in A$. But this contradicts the assumption that $\gamma \circ \varphi$ is α-complete. For then $\gamma \circ \varphi(c) = \bigvee^N \gamma \circ \varphi[A]$. $\square$

Henceforth, $L \subseteq M$ denotes that L is a $\mathcal{W}$ subspace of M (i.e., L is a vector lattice subspace of M, and L has the same distinguished weak unit as M). $L \subseteq^\alpha M$ denotes that L is a $\mathcal{W}$ subspace of M, and the inclusion of L into M is an α-complete morphism. As usual, we reserve $\subset$ for ordinary set inclusion.

Definition 2.3. Let $L \subseteq M$. L is said to be *α-dense* in M if for each $b \in M^+$ there is an $A \subset L$ with $|A| < \alpha$ such that $b = \bigvee^M A$. L is said to be *α-jam-dense* in M if for each $c \in M^+$ there are $A, B \subset L$ with $|A|, |B| < \alpha$ such that $c = \bigwedge^M B = \bigvee^M A$. (Or, $0 = \bigwedge^M (B - A)$ and $B \geq c \geq A$.)

An embedding $\gamma : L \hookrightarrow M$ is said to be α-dense (α-jam-dense) if $\gamma[L]$ is α-dense (α-jam-dense) in M. Later we will see (Proposition 2.8) that α-dense embeddings are epic in $\mathcal{W}(\alpha)$. Clearly, α-jam-density implies α-density, but the converse is not true. For $C(X)$ is ω_1-dense, but not ω_1-jam-dense, in $D(X)$ whenever $D(X)$ is indeed a $\mathcal{W}$ object. (See [17] and Section 3 here.)

Proposition 2.4. *Suppose L and M are $\mathcal{S}$ objects. Then L is α-dense in M if and only if L is α-jam-dense in M.*

Proof. The sufficiency is clear. On the other hand, let L be α-dense in M. For each $c \in M^+$ there is $A \subset L$ with $|A| < \alpha$ such that $c = \bigvee^M A$. If u is the strong unit in L and M, there is an $n \in \mathbb{N}$ such that $nu > c$. So $nu - c \in M^+$, and there is a $B \subset L$ with $|B| < \alpha$ such that $nu - c = \bigvee^M B$. It follows that $c = \bigwedge^M \{nu - b : b \in B\}$. Therefore L is α-jam-dense in M. $\square$

The following propositons are well known [6, 32, 26].

Proposition 2.5. *L is ∞-dense in M if and only if for each $b \in M^+$ there is an $a \in L^+$ such that $b > a > 0$.*

Proposition 2.6. *If L is ∞-dense in M, then L is α-completely embedded in M for all $\alpha < \infty$ (i.e., $L \subseteq^\infty M$). Therefore if L is α-dense in M, then $L \subseteq^\alpha M$.*

Corollary 2.7. *If $\varphi : L \hookrightarrow M$ is α-dense, then φ is ∞-complete (and hence α-complete for all $\alpha < \infty$).*

Proof. L is isomorphic to $\varphi[L]$, and $\varphi[L] \subseteq^\infty M$ (Proposition 2.6). Therefore φ is ∞-complete because it is the composition of two ∞-complete morphisms. □

We may assume that $L \subseteq^\infty M$ whenever there is an ∞-dense embedding of L into M.

Proposition 2.8. *If $\varphi : L \hookrightarrow M$ is α-dense, then φ is epic in $\mathscr{W}(\alpha)$.*

Proof. By Corollary 2.7, φ is α-complete and therefore a morphism in $\mathscr{W}(\alpha)$. Let $\gamma_i : M \hookrightarrow N$, with $i = 1, 2$, be α-complete and suppose $\gamma_1 \circ \varphi = \gamma_2 \circ \varphi$. Let $b \in M$. Without loss of generality we may assume that $b \in M^+$. Therefore $b = \bigvee^M \varphi[A]$ for some $A \subset L$ with $|A| < \alpha$. Hence $\gamma_1(b) = \gamma_1(\bigvee^M \varphi[A]) = \bigvee^N \gamma_1 \circ \varphi[A] = \bigvee^N \gamma_2 \circ \varphi[A] = \gamma_2(\bigvee^M \varphi[A]) = \gamma_2(b)$ because $\gamma_1 \circ \varphi(a) = \gamma_2 \circ \varphi(a)$ for all $a \in A$. □

Proposition 2.2 says that if the second factor of an α-complete morphism is injective, then the first factor of that α-complete morphism is also α-complete. Unfortunately the second factor of an α-complete morphism need not be α-complete; however:

Lemma 2.9 (see [26, 23.2)]. *Let $\gamma : L \hookrightarrow M$ be α-dense and $\varphi : M \to N$. If $\varphi \circ \gamma$ is α-complete, then φ is α-complete.*

Proof. We may assume that L is α-dense in M. Suppose φ is not α-complete. Then by Proposition 2.1(b), there is for each $c \in M^+$ a $B_c \subset M^+$ with $|B_c| < \alpha$ and $c = \bigvee^M B_c$, but $\varphi(c) \neq \bigvee^N \varphi[B_c]$. Therefore there is a $d_c \in N^+$ such that $\varphi(c) > d_c > \varphi(b)$ for all $b \in B_c$. Pick $a \in L^+$ (hence $a \in M^+$) and consider B_a as described above. For each $b_i \in B_a$ there is an $A_i \subset L$ with $|A_i| < \alpha$ and $b_i = \bigvee^M A_i$ because L is α-dense in M. Moreover, $a = \bigvee^M \bigcup_i A_i = \bigvee^L \bigcup_i A_i$ because $a \in L^+$, and $\bigcup_i A_i \subset L$. We claim that $\varphi|_L$ is not α-complete, i.e., we show that $\varphi(a) \neq \bigvee^N \varphi[\bigcup_i A_i]$ ($|\bigcup_i A_i| < \alpha$). As discussed above, there is a $d_a \in N^+$ such that $\varphi(a) > d_a > \varphi(b_i)$ for all $b_i \in B_a$. From this we conclude that $\varphi(a) > d_a > \varphi(a_i)$ for all $a_i \in A_i$ and for all i. Therefore $\varphi(a) \neq \bigvee^N \varphi[\bigcup_i A_i]$. □

Let $L \subseteq M$ and $b \in M^+$. The smallest $\mathcal{W}$ subspace of M that contains L and b is the smallest subset of M that contains L and b and is closed under the vector lattice operations. Using distributive laws in [6, 7, 10] we have:

Proposition 2.10. *Let $L \subseteq M$ and $b \in M$. Then $\langle L, b\rangle^M$, the smallest $\mathcal{W}$ subspace of M that contains L and b, is $\{\bigwedge_{i=1}^n \bigvee_{j=1}^k a_{ij} + r_{ij}b\colon a_{ij} \in L,\ r_{ij} \in \mathbb{R},\ n, k \in \mathbb{N}\}$.*

Note, the superscript ^{M} in $\langle L, b\rangle^M$ is omitted when the context is clear.

Lemma 2.11. *Let $L \subseteq M$, $A, B \subset L$, $|A|, |B| < \alpha$, $b \in M^+$, and $b = \bigvee^M A = \bigwedge^M B$. Then L is α-jam-dense in $\langle L, b\rangle$.*

Proof. Let $c \in \langle L, b\rangle^+$. Then $c = \bigwedge_{i=1}^n \bigvee_{j=1}^k (a_{ij} + r_{ij}b)$ for some $a_{ij} \in L$, $r_{ij} \in \mathbb{R}$, and $n, k \in \mathbb{N}$. For each (i, j) let $d_{ij} = a_{ij} + r_{ij}b = a_{ij} + r_{ij} \bigvee^M A = a_{ij} + r_{ij} \bigwedge^M B$. We have

$$c = \bigwedge_{i=1}^{n} \bigvee_{j=1}^{k} d_{ij}.$$

Now for $r_{ij} > 0$,

$$d_{ij} = \bigvee{}^M \{a_{ij} + r_{ij}a\colon a \in A\} = \bigwedge{}^M \{a_{ij} + r_{ij}p\colon p \in B\},$$

and for $r_{ij} \leqslant 0$,

$$d_{ij} = \bigwedge{}^M \{a_{ij} + r_{ij}a\colon a \in A\} = \bigvee{}^M \{a_{ij} + r_{ij}p\colon p \in B\}.$$

For each (i, j) define the sets

$$C_{ij} = \begin{cases} \{a_{ij} + r_{ij}a\colon a \in A\}, & \text{if } r_{ij} > 0, \\ \{a_{ij} + r_{ij}p\colon p \in B\}, & \text{if } r_{ij} \leqslant 0, \end{cases}$$

$$D_{ij} = \begin{cases} \{a_{ij} + r_{ij}p\colon p \in B\}, & \text{if } r_{ij} > 0, \\ \{a_{ij} + r_{ij}a\colon a \in A\}, & \text{if } r_{ij} \leqslant 0. \end{cases}$$

Then $d_{ij} = \bigvee^M C_{ij} = \bigwedge^M D_{ij}$.

It follows that $c = \bigwedge_{i=1}^n \bigvee_{j=1}^k (\bigvee^M C_{ij}) = \bigwedge_{i=1}^n \bigvee_{j=1}^k (\bigwedge^M D_{ij})$, and $c = \bigvee^M C' = \bigwedge^M D'$ with $C', D' \subset L$ and $|C'|, |D'| < \alpha$. Since $c \in \langle L, b\rangle$, $c = \bigvee^{\langle L,b\rangle} C' = \bigwedge^{\langle L,b\rangle} D'$. Hence L is α-jam-dense in $\langle L, b\rangle$. $\square$

In light of Lemma 2.11 one may ask: if $L \subseteq M$, $b \in M^+$, $A \subset L$, $|A| < \alpha$, and $b = \bigvee^M A$, then is L α-dense in $\langle L, b\rangle$? The answer is "no" (Example 2.12) even if both L and M are $\mathcal{S}$ objects.

Example 2.12. Let $X \in |\mathrm{Comp}|$ be extremally disconnected [11]. Suppose U is open, but not clopen in X. Let $L(X)$ be the $\mathcal{W}$ object of locally constant real-valued functions on X [10, 32]. Consider $L(X) \subseteq \mathbb{R}^X$ (**1** is the strong unit). The characteristics function of U, χ_U, is not an element of $L(X)$, and clearly $\chi_U = \bigvee^{\mathbb{R}^X} \{b \in L(X)\colon b < \chi_U\}$. (Actually, χ_U is the sup of a subset of $L(X)$ whose cardinality is equal to the weight of X.) We claim that $L(X)$ is not ∞-dense in $\langle L(X), \chi_U\rangle$.

Let $K = \bar{U} - U$. Since $\mathbf{1} - \chi_U = \chi_{X \setminus U} \in \langle L(X), \chi_U \rangle^+$, $\chi_K = \chi_{X \setminus U} \wedge \chi_{\bar{U}} \in \langle L(X), \chi_U \rangle^+$. However, since K is nowhere dense, there is no $b \in L(X)^+$ such that $\chi_K > b > 0$. Therefore, by Proposition 2.5, $L(X)$ is not ∞-dense in $\langle L(X), \chi_U \rangle$.

If $L \subseteq N \subseteq M$ and L is α-dense (α-jam-dense) in N, N is an α-dense (α-jam-dense) extension of L in M. Below we show there are maximal, sometimes maximum, such extensions.

Proposition 2.13. *Let $L \subseteq M$ and P be a family of subspaces of M. View P as a poset ordered by set inclusion and suppose C is a chain in P.*

(a) *If $P = \{L_i \subseteq M: L \subseteq^\alpha L_i\}$, then $\bigcup C \in P$, i.e., $L \subseteq^\alpha \bigcup C$.*

(b) *If $P = \{L_i \subseteq M: L$ is α-dense in $L_i\}$, then $\bigcup C \in P$, i.e., L is α-dense in $\bigcup C$.*

(c) *If $P = \{L_i \subseteq M: L$ is α-jam-dense in $L_i\}$, then $\bigcup C \in P$, i.e., L is α-jam-dense in $\bigcup C$.*

Proof. Obviously an increasing union of $\mathscr{W}$ subobjects in M is a $\mathscr{W}$ subobject of M.

(a) Suppose L is not α-completely embedded in $\bigcup C$. Then there is a $c \in L_i^+ \in C$ and an $A \subset L$ with $|A| < \alpha$ and $c = \bigvee^{L_i} A$ such that $c \neq \bigvee^{\bigcup C} A$. So there is a $b \in \bigcup C$ such that $c > b > a$ for all $a \in A$. Since $b \in L_i$ for some $L_i \in C$, $c \neq \bigvee^{L_i} A$. But this is a contradiction since $L \subseteq^\alpha L_i$.

(b) Since $L \subseteq^\alpha L_i$ (Proposition 2.6) we have by (a) here that $L \subseteq^\alpha \bigcup C$. Let $b \in \bigcup C^+$. Then $b \in L_i^+ \in C$ for some i, and there is an $A \subset L$ with $|A| < \alpha$ and $b = \bigvee^{L_i} A$. By Lemma 2.9, $L_i \subseteq^\alpha \bigcup C$, hence $b = \bigvee^{\bigcup C} A$. Therefore L is α-dense in $\bigcup C$.

(c) As in (b), $L \subseteq^\alpha \bigcup C$. Let $c \in \bigcup C^+$. Then $c \in L_i^+ \in C$ for some i, and there are A, $B \subset L$ with $|A|$, $|B| < \alpha$ such that $c = \bigwedge^{L_i} B = \bigvee^{L_i} A$. By Lemma 2.9, $L_i \subseteq^\alpha \bigcup C$, and it follows from Proposition 2.1 that $c = \bigvee^{\bigcup C} A = \bigwedge^{\bigcup C} B$ and L is α-jam-dense in $\bigcup C$. $\square$

Lemma 2.14. *Let $L \subseteq M$. There are L_1, L_2, and L_3 with $L \subseteq^\alpha L_i \subseteq M$ such that:*

(a) *L_i is maximal for the property of containing L as an α-completely embedded subspace.*

(b) *L_2 is maximal for the property of containing L as an α-dense subspace.*

(c) *L_3 is maximal for the property of containing L as an α-jam-dense subspace.*

Proof. Proposition 2.13 allows us to invoke Zorn's lemma in each of the above cases. $\square$

If in Lemma 2.14, $L \subseteq^\alpha M$, we have:

Theorem 2.15. *Let $L \subseteq^\alpha M$. There is a subspace of M, denoted $[L]_\alpha^M$, such that $[L]_\alpha^M$ is maximum (among the subspaces of M) for the property of containing L as an α-jam-dense subspace. If L and M are $\mathscr{S}$ objects, then $[L]_\alpha^M$ is maximum for the property of containing L as an α-dense subspace (Proposition 2.4). Moreover, if M is α-Dedekind complete, then $[L]_\alpha^M$ is also α-Dedekind complete.*

Proof. Suppose L_1, L_2, $L_1 \neq L_2$, are maximal for the property of containing L as an α-jam-subspace. Then there is a $b \in L_1^+ \setminus L_2^+$, and there are $A, B \subset L_1$ with $|A|, |B| < \alpha$ such that $b = \bigvee^{L_1} A = \bigwedge^{L_1} B$. Since $L_1 \subseteq^\alpha M$ (Proposition 2.6), $b = \bigvee^M A = \bigwedge^M B$. Then by Example 2.12, L_2 is α-jam-dense in $\langle L_2, b \rangle$. But this contradicts the maximality of L_2.

Now suppose that M is α-Dedekind complete, but $[L]_\alpha^M$ is not. Then there are $A, B \subset [L]_\alpha^{M+}$ with $|A| < \alpha$, $|B| < \alpha$, $B > A$, and $0 = \bigwedge^{[L]_\alpha^M} (B - A)$ for which there does not exist a $c \in [L]_\alpha^M$ satisfying $B > c > A$. $[L]_\alpha^M \subseteq^\alpha M$, so $0 = \bigwedge^M (B - A)$ (Proposition 2.1), and since M is α-Dedekind complete, there is a $c \in M$ such that $c = \bigvee^M A = \bigwedge^M B$. Hence by Lemma 2.11, $[L]_\alpha^M$ is α-jam-dense in $\langle [L]_\alpha^M, c \rangle$, but this contradicts the maximality of $[L]_\alpha^M$. $\square$

We now give the main result of this section.

Theorem 2.16. *Let $\gamma : L \hookrightarrow M$ be an α-jam-dense embedding and suppose $\varphi : L \to N$ is α-complete with N α-Dedekind complete. Then there is a unique $\bar{\varphi} : M \to N$ such that $\bar{\varphi} \circ \gamma = \varphi$. $\bar{\varphi}$ is necessarily α-complete. Moreover, if L and M are $\mathcal{S}$ objects, then the same lifting result holds when γ is an α-dense embedding* (*Proposition* 2.4).

Proof. We may assume that L is an α-jam-dense subspace of M. Also, if $\bar{\varphi}$ exists, it is α-complete (Lemma 2.9), and hence unique (Proposition 2.8).

We define $\bar{\varphi}$. Let $b \in M$. Without loss of generality we can assume that $b \in M^+$. There are $A, B \subset L$ with $|A|, |B| < \alpha$, with $b = \bigvee^M A = \bigwedge^M B$. So $0 = \bigwedge^M (B - A)$. Therefore by Proposition 2.1, $0 = \varphi(0) = \bigwedge^N (\varphi[B] - \varphi[A])$. From this it follows, since N is α-Dedekind complete, that there is a $c \in N$ such that $c = \bigvee^N \varphi[A] = \bigwedge^N \varphi[B]$. Define $\bar{\varphi}(b) = c$. Clearly $\bar{\varphi}$ is an assignment that extends φ, and it is not hard to see that $\bar{\varphi}$ is a $\mathcal{W}$ morphism if, as is shown below, it is well defined.

Let $d \in M$ and suppose $d = \bigvee^M A_1 = \bigwedge^M B_1 = \bigvee^M A_2 = \bigwedge^M B_2$ for A_i, B_i as above. It follows that there are $c_1, c_2 \in N$ such that

$$c_1 = \bigvee^N \varphi[A_1] = \bigwedge^N \varphi[B_1], \quad \text{and} \quad c_2 = \bigvee^N \varphi[A_2] = \bigwedge^N \varphi[B_2].$$

To see that $\bar{\varphi}$ is well defined it suffices to see that $c_1 = c_2$.

To this end, we claim that $c_1 \geq \varphi(a')$ for all $a' \in A_2$, and $c_1 \leq \varphi(b')$ for all $b' \in B_2$. From this it follows that $c_1 = c_2 = \bigvee^N \varphi[A_2] = \bigwedge^N \varphi[B_2]$.

For each $a' \in A_2 \subset L$ we have

$$\bigvee^M \{a' \wedge a : a \in A_1\} = a' \wedge \bigvee^M A_1 = a' \wedge d = a'.$$

Therefore $\bigvee^L \{a' \wedge a : a \in A_1\} = a'$. Since φ is α-complete, we have that

$$\begin{aligned} \varphi(a') &= \bigvee^N \{\varphi(a') \wedge \varphi(a) : a \in A_1\} \\ &= \varphi(a') \wedge \bigvee^N \varphi[A_1] = \varphi(a') \wedge c_1. \end{aligned}$$

Therefore $c_1 \geq \varphi(a')$ for all $a' \in A_2$. Similarly for each $b' \in B_2 \subset L$ we have,

$$\bigwedge^M \{b' \vee b: b \in B_1\} = b' \vee \bigwedge^M B_1 = b' \vee d = b'.$$

Hence $\bigwedge^L \{b' \vee b: b \in B_1\} = b'$. It follows that

$$\begin{aligned}\varphi(b') &= \bigwedge^N \{\varphi(b') \vee \varphi(b): b \in B_1\} \\ &= \varphi(b') \vee \bigwedge^N \varphi[B_1] = \varphi(b') \vee c_1.\end{aligned}$$

Therefore $\varphi(b') \geq c_1$ for all $b' \in B_2$. □

The next theorem is well known and it tells us that each L has an essentially unique Dedekind (∞-Dedekind complete) completion [32].

Theorem 2.17. *Each L has an essentially unique pair $(\sigma, \bar{L})$ where $\bar{L}$ is Dedekind complete and σ is an ∞-jam-dense embedding. That is, for each $b \in \bar{L}^+$,*

$$b = \bigvee^{\bar{L}} \{\sigma(a): \sigma(a) \leq b\} = \bigwedge^{\bar{L}} \{\sigma(a): \sigma(a) \geq b\}.$$

$(\sigma, \bar{L})$ is called the Dedekind completion of L.

When we say $(\sigma, \bar{L})$ is *essentially unique*, we mean that if given another pair $(\sigma', \bar{L}')$ with the same properties (as in Theorem 2.17), there is an isomorphism, $\tau: \bar{L} \to \bar{L}'$, such that $\sigma' = \tau \circ \sigma$. Because of this, we will suppress reference to the embedding σ. That is, we can consider L to be an ∞-jam-dense subspace of a unique Dedekind complete $\mathcal{W}$ object $\bar{L}$; and we say that $\bar{L}$ "is" the Dedekind completion of L. We begin to generalize Theorem 2.17 to arbitrary α. For an L, we say that (φ, M) is an α-jam-dense (α-dense) extension of L if $\varphi: L \hookrightarrow M$ is α-jam-dense (α-dense).

Proposition 2.18. *An α-jam-dense extension, (φ, M), of L, with M α-Dedekind complete is essentially unique.*

Proof. Apply Theorem 2.16. □

Recall in an abstract category $\mathcal{B}$, a full subcategory $\mathcal{A}$ is called *epireflective* if for each $B \in |\mathcal{B}|$ there is an $A_B \in |\mathcal{A}|$ and an epimorphism $e: B \to A_B$ such that for any morphism $f: B \to A$ with $A \in |\mathcal{A}|$ there exists an $\bar{f}: A_B \to A$ such that $f = \bar{f} \circ e$. (e, A_B) is called the $\mathcal{A}$ *epireflection* of B.

We claim that, in $\mathcal{W}(\alpha)$, the full subcategory of α-Dedekind complete objects is epireflective. Because of the lifting theorem (Theorem 2.16), we need only produce, for each L, an α-jam-dense extension, call it $(i, \hat{L}_\alpha)$, with $\hat{L}_\alpha$ α-Dedekind complete. This is easily done. Recall that L is ∞-jam-dense in $\bar{L}$. Define $\hat{L}_\alpha = [L]_\alpha^{\bar{L}}$ to be the maximum α-jam-dense extension of L in $\bar{L}$ (Theorem 2.15). $\hat{L}_\alpha$ is α-Dedekind complete because $\bar{L}$ is α-Dedekind complete for all $\alpha < \infty$. Hence $(i, \hat{L}_\alpha)$ is the α-Dedekind complete epireflection of L where i is the inclusion of L into $\hat{L}_\alpha$. (i is epic by Proposition 2.8.) Note, $\hat{L}_\infty = \bar{L}$. We summarize below.

Theorem 2.19. *In $\mathcal{W}(\alpha)$, the full subcategory of α-Dedekind complete objects is epireflective, and for each L, $(i, \hat{L}_\alpha)$ is the α-Dedekind complete epireflection of L.*

3. The Yosida functor

In this section, $X, Y, Z \in |\mathrm{Comp}|$; f, g, and h denote continuous functions.

The functor Y works very much like the Stone functor from Boolean algebras to Boolean spaces. $Y(L)$, like the Stone space of a Boolean algebra, is a maximal ideal space. The elements of $Y(L)$ are ideals of L that are maximal for the property of not containing the weak unit. If the weak unit is a strong unit, then these ideals are the actual maximal ideals of L. The topology on this space is the hull-kernel topology. In fact, if we view a Boolean algebra, B, as an $\mathcal{S}$ object (i.e., consider $L(S(B))$, the locally constant real-valued functions on the Stone space, $S(B)$, of B), the Yosida functor can be thought of as an extension of the Stone functor.

The following discussion comes from [2]. Let $C(X)$ be the $\mathcal{S}$ object of real-valued continuous functions on X. The strong unit of $C(X)$ will always be taken to be the constant function $\mathbf{1}$. Let $D(X)$ be the set of extended real-valued continuous functions, $f: X \to [-\infty, +\infty]$, for which $f^{-1}(\mathbb{R})$ is dense in X. In the pointwise order, $D(X)$ is a lattice, but usually fails to be a vector space. For $f, g, h \in D(X)$, we say "$f+g=h$ in $D(X)$" if $f(x)+g(x)=h(x)$ when $x \in f^{-1}(\mathbb{R}) \cap g^{-1}(\mathbb{R}) \cap h^{-1}(\mathbb{R})$ (which is a dense set in X). It may well happen that, for particular $f, g \in D(X)$, there is no $h \in D(X)$ with $f+g=h$ in $D(X)$ (e.g., take $X=[-\infty, +\infty]$, f the obvious extension of $x+\sin x$, and g the extension of $-x$). However, it may well happen that a subset $L \subset D(X)$ has the property that for all $f, g \in L$ there is an $h \in D(X)$ with $f+g=h$ in $D(X)$; if L is also a vector lattice under the pointwise operation in $D(X)$ and the constant function, $\mathbf{1}$, is in L, then we say "$(L, \mathbf{1})$ (or just L) is a $\mathcal{W}$ object in $D(X)$" (e.g., $C(X)$ is a vector lattice, indeed, a $\mathcal{W}$ object, in $D(X)$). If X has the property that each dense cozero set is C^*-embedded [11], then X is called *ω_1-quasi-F* (or just *quasi-F*). See Definition 1.5 here and [5, 9, 18]. If X is ω_1-quasi-F, then $(D(X), \mathbf{1}) \in |\mathcal{W}|$. See [17].

Theorem 3.1 [15]. (a) *There is a $\mathcal{W}$ isomorphism, $\hat{}: L \to \hat{L} \subset D(Y(L))$, onto a $\mathcal{W}$ object, $\hat{L}$ in $D(Y(L))$, with weak unit $\hat{w}_L = \mathbf{1}$, and $\hat{L}$ separates the points of $Y(L)$.*

(b) *If L' is a $\mathcal{W}$ object in $D(X)$ which separates the points of X, and for $a \in L$, if $a \mapsto a'$ is a $\mathcal{W}$ isomorphism from L to L', then there is a homeomorphism $f: X \to Y(L)$ such that $a' = \hat{a} \circ f$ for all $a \in L$.*

Theorem 3.1(b) is used to recognize Yosida representations.

Let $L^* = (w_L)$ be the principal ideal generated by w_L in L. $\hat{L}^*$ is a $\mathcal{W}$ object (indeed, an $\mathcal{S}$ object) in $D(Y(L))$ and consists of all $\hat{a} \in \hat{L}$ which are bounded. Note that if L is an $\mathcal{S}$ object, i.e., if w_L is a strong unit, then $L = L^*$.

Corollary 3.2. (a) $Y(L^*) = Y(L)$.

(b) $Y(C(X)) = X$.

(c) *$\hat{L}^*$ is an $\mathcal{S}$ subspace of $C(Y(L))$.*

(d) *If L is an $\mathcal{S}$ object, then $\hat{L} \subseteq C(Y(L))$.*

Proof. (a) In $D(Y(L))$, $\hat{L}^*$ satisfies Theorem 3.1(b).

(b) In $D(X)$, $C(X)$ satisfies Theorem 3.1(b).

(c) $\hat{a} \in \hat{L}^*$ implies $\hat{a}^{-1}(\{\pm\infty\}) = \emptyset$, hence $\hat{a} \in C(Y(L))$.

(d) If $L \in |\mathcal{S}|$, then $L = L^*$. $\square$

Let us look at a couple of simple examples. Since $\mathbb{R}^X = C(D_X)$, where D_X is the set X with the discrete topology, we can seen from Corollary 3.2(a), (b) that $Y(\mathbb{R}^X) = \beta D_X$ ($C^*(D_X) = C(\beta D_X)$).

Let X be zero dimensional. Since $L(X) \subseteq C(X)$ and $L(X)$ (Example 2.12) separates the points of X, we can see from Theorem 3.1(b) and Corollary 3.2(b) that $Y(L(X)) = X$.

Theorem 3.3 [15]. *Let $\varphi_i : L \to M$ for $i = 1, 2$.*

(a) *There is a unique continuous function, $Y(\varphi_1) : Y(M) \to Y(L)$, such that $\varphi(a)\hat{} = \hat{a} \circ Y(\varphi_1)$ for all $a \in L$.*

(b) *Y is a faithful functor, i.e., if $\varphi_1 \neq \varphi_2$, then $Y(\varphi_1) \neq Y(\varphi_2)$.*

(c) *φ_1 is one-to-one if and only if $Y(\varphi_1)$ is onto, and if φ_1 is onto, then $Y(\varphi_1)$ is one-to-one.*

Henceforth, L and $\hat{L}$ are identified.

We will consider $a \in L$ as an extended real-valued function on $Y(L)$, and if L is an $\mathcal{S}$ object, we will consider $a \in L$ as a real-valued function on $Y(L)$. See Corollary 3.2. In general, $Y(\varphi)$ being one-to-one does not imply that φ is onto: if φ is the inclusion of $L(X)$ (Example 2.12) into $C(X)$, then $Y(\varphi) : X = Y(C(X)) \to Y(L(X)) = X$ is id_X. However:

Proposition 3.4. *Let $\varphi : C(X) \to M$ and $M \in |\mathcal{S}|$. Then $Y(\varphi) : Y(M) \to X$ is one-to-one if and only if φ is onto.*

Proof. For the sufficiency apply Theorem 3.3(c).

On the other hand, if $Y(\varphi)$ is one-to-one, then $Y(M)$ is homeomorphic (via $Y(\varphi)$) to a closed subspace K of X. Hence there is a homeomorphism, $Y(\varphi)^{-1} : K \to Y(M)$. Let $b \in M$. Since $M \subseteq C(Y(M))$, we have that $b \circ Y(\varphi)^{-1} \in C(K)$. Therefore, there is a continuous extension, $\bar{b} \in C(X)$, of $b \circ Y(\varphi)^{-1}$ to all of X. Then, by Theorem 3.3(a), $\varphi(\bar{b}) = \bar{b} \circ Y(\varphi) = b$. $\square$

The next proposition is straightforward.

Proposition 3.5. *Let $f: X \to Y$. Define $f': C(Y) \to C(X)$ by $f'(g) = g \circ f$ for $g \in C(Y)$* [11, Ch. 10]. *Then f' is a $\mathscr{W}$ morphism and $Y(f') = f$.*

Theorem 3.6 (Banach-Stone). *A function $f: X \to Y$ is a homeomorphism if and only if $f': C(Y) \to C(X)$ is an isomorphism in $\mathscr{W}$.*

Proof. $Y(f') = f$. Apply Theorem 3.3(c) and Proposition 3.4. □

We begin a discussion of the topological category α-SpFi. See Definition 1.4 here, and [4, 5, 20]. A topological space, K, is called α-Lindelöf if any open cover of K has a subcover of cardinality less than α. A subset of X is called an F_α-set in X if it is the union of fewer than α many closed subsets of X. Clearly every F_α-set in X is α-Lindelöf.

Definition 3.7. Let $\operatorname{Coz}(X) = \{f^{-1}(\mathbb{R}\setminus\{0\}): f \in C(X)\}$. A subset $V \subset X$ is said to be an *α-cozero* set if $V = \bigcup \{U_i: i \in I, |I| < \alpha, U_i \in \operatorname{Coz}(X)\}$. Note that an ω_1-cozero set is a cozero set. Recall that by "$|I| < \infty$" we mean that "$|I|$ is unrestricted", so that every open set is an ∞-cozero set. We denote the collection of α-cozero sets of X by $\operatorname{Coz}_\alpha(X)$. Let $\mathscr{G}_\alpha(X)$ be the filter generated by the dense members of $\operatorname{Coz}_\alpha(X)$. $\mathscr{G}_\infty(X)$ will denote the filter generated by the dense open sets. Finally let $\mathscr{G}_\alpha(X)_\delta$ be the filter generated by

$$\{\bigcap G_n: n \in \mathbb{N},\ G_n \in \mathscr{G}_\alpha(X)\}.$$

Note the Baire category theorem implies that $\mathscr{G}_\alpha(X)_\delta$ is a filter of dense sets. Moreover, when ω_0 is replaced by α [8, 9.7, 9.8, and 9.10(a)] show that the members of $\mathscr{G}_\alpha(X)_\delta$ are α-Lindelöf.

α-SpFi is actually a full subcategory of the more general category of *spaces with filters*, denoted SpFi, which has for objects pairs of the form $(X, \mathscr{F})$ where $\mathscr{F}$ is a filter base of dense subsets of X. A SpFi morphism $f: (X, \mathscr{F}) \to (Y, \mathscr{H})$ is a continuous function from X to Y that inversely preserves the elements of the filter bases, i.e., $f^{-1}(H) \in \mathscr{F}$ for all $H \in \mathscr{H}$.

Proposition 3.8. *Let $f: X \to Y$. The following are equivalent.*

(a) *f is an α-*SpFi *morphism.*

(b) *$f^{-1}(G)$ is dense in X for each $G \in \mathscr{G}_\alpha(Y)$, i.e., $f: (X, \mathscr{G}_\alpha(X)) \to (Y, \mathscr{G}_\alpha(Y))$ is a* SpFi *morphism.*

(c) *$f^{-1}(H)$ is dense in X for each $H \in \mathscr{G}_\alpha(Y)_\delta$, i.e., $f: (X, \mathscr{G}_\alpha(X)_\delta) \to (Y, \mathscr{G}_\alpha(Y)_\delta)$ is a* SpFi *morphism.*

Proof. The equivalence of (b) and (c) is clear. Also (a)$\Rightarrow$(b) is trivial because each $G \in \mathscr{G}_\alpha(Y)$ is an F_α-set in Y and hence α-Lindelöf. For (b)$\Rightarrow$(a) we use [4, 1.6]: if f is as in (b), U is regular closed in X, and $G \in \mathscr{G}_\alpha(Y)$, then $f[U] \cap G$ is dense in $f[U]$.

So now let K be dense and α-Lindelöf in Y and suppose $f^{-1}(K)$ is not dense in X. Then there is a nonempty regular closed set U in X such that $U \cap f^{-1}(K) = \emptyset$. Therefore $f[U] \cap K = \emptyset$. It follows that there exists an α-cozero set V in Y such that $K \subset V$ and $V \cap f[U] = \emptyset$. But this contradicts [4, 1.6]. □

The next three lemmas (Lemmas 3.9–3.11) provide a fundamental link between $\mathscr{W}(\alpha)$ and α-SpFi. (In regard to Lemmas 3.9 and 3.10, the essential ideas and content come from [3, 4.1 and 4.2].)

Lemma 3.9. *Let $b \in L^+$, $A \subset L$, $|A| < \alpha$, $r \in \mathbb{R}$ and let $P_b = \{x \in Y(L): b(x) = \bigvee^{\mathbb{R}} \{a(x): a \in A\}\}$.*

(a) *$b = \bigvee^L A$ if and only if $P_b \in \mathscr{G}_\alpha(Y(L))_\delta$.*

(b) *If $r = \bigvee^L A$, then $\bigcup \{\mathrm{coz}(a): a \in A\} \in \mathscr{G}_\alpha(Y(L))$.*

(c) *If $G \in \mathscr{G}_\alpha(Y(L))$, then for each $r \in \mathbb{R}$ there is a family $A_r \subset L^+$ with $|A_r| < \alpha$ such that $G = \bigcup \{\mathrm{coz}(a): a \in A_r\}$ and $r = \bigvee^L A_r$.*

Proof. (a) If $P_b \in \mathscr{G}_\alpha(Y(L))_\delta$, then P_b is dense in $Y(L)$, hence $b = \bigvee^L A$. On the other hand, suppose $b = \bigvee^L A$. For each $n \in \mathbb{N}$ let $U_n = \{x \in Y(L): b(x) - a(x) < n^{-1}$ for some $a \in A\}$. It is not hard to see that each $U_n \in \mathrm{Coz}_\alpha(Y(L))$ and $P_b = \bigcap_n U_n$. If each U_n is dense in $Y(L)$, we are finished. Suppose not, then there is an open set $V \subset Y(L)$ for which $b(x) - a(x) \geq n^{-1}$ for all $x \in V$ and for all $a \in A$. Take another open set, $V' \subset V$, with $\bar{V}' \subset V$. Since L separates the points of $Y(L)$ (Theorem 3.1), there is a $c \in L^+$ such that $c[Y(L) \setminus V] = 0$, $c[\bar{V}'] = n^{-1}$, and $0 \leq c \leq n^{-1}$. It follows that $b - c > a$ for all $a \in A$, but this contradicts $b = \bigvee^L A$.

(b) If $r = \bigvee^L A$, then by (a), P_r is dense in $Y(L)$. Since $P_r \subset \bigcup \{\mathrm{coz}(a): a \in A\}$, it follows that $\bigcup \{\mathrm{coz}(a): a \in A\} \in \mathscr{G}_\alpha(Y(L))$.

(c) Let $G \in \mathscr{G}_\alpha(Y(L))$ and $x \in G$. There is an $a_x \in L^+$, and a neighborhood V_x of x such that $\bar{V}_x \subset G$, $a_x[\bar{V}_x] = r$, $a_x[X \setminus G] = 0$, and $0 < a_x \leq r$. Since G is α-Lindelöf, there is a subfamily $\{V_{x'}\}$ of cardinality less than α with $G = \bigcup \{V_{x'}\}$. Therefore since $V_{x'} \subset \mathrm{coz}(a_{x'}) \subset G$, we have $G = \bigcup \{\mathrm{coz}(a_{x'})\}$. Clearly, $r = \bigvee^L \{a_{x'}\}$. Take $A_r = \{a_{x'}\}$. □

Lemma 3.10. *$\varphi : L \to M$ is α-complete if and only if $Y(\varphi): Y(M) \to Y(L)$ is an α-*SpFi *morphism.*

Proof. Let φ be α-complete. For $G \in \mathscr{G}_\alpha(Y(L))$ take an $A_1 \subset L^+$ as in Lemma 3.9(c). Since φ is α-complete, and $\varphi(a) = a \circ Y(\varphi)$ for all $a \in A$ (Theorem 3.3(a)), it follows that $\mathbf{1} = \varphi(\mathbf{1}) = \bigvee^M \{\varphi(a): a \in A\} = \bigvee^M \{a \circ Y(\varphi): a \in A\}$. Applying Lemma 3.9(a) we know that $P_1 = \{x \in Y(M): 1 = \bigvee^{\mathbb{R}}_A a \circ Y(\varphi)(x)\}$ is dense in $Y(M)$. Now, because

$x \in P_1$ implies that $x \in Y(\varphi)^{-1}(\mathrm{coz}(a))$ for some $a \in A$, we have $P_1 \subset Y(\varphi)^{-1}(\bigcup_A \mathrm{coz}(a)) = Y(\varphi)^{-1}(G)$. So $Y(\varphi)^{-1}(G)$ is dense in $Y(M)$. Therefore $Y(\varphi) \in \alpha$-SpFi (Proposition 3.8(b)).

On the other hand, suppose $Y(\varphi)$ is an α-SpFi morphism and let $\mathbf{1} = \bigvee^L A$ with $|A| < \alpha$. Applying Lemma 3.9 we know that $P_1 = \{x \in Y(L): 1 = \bigvee_A a(x)\} \in \mathscr{G}_\alpha(Y(L))_\delta$. For $x \in Y(M)$, if $Y(\varphi)(x) \in P_1$, then $1 = \bigvee_A a \circ Y(\varphi)(x) = \bigvee_A \varphi(a)(x)$. Hence $Y(\varphi)^{-1}(P_1) \subset \{y \in Y(M): 1 = \bigvee_A \varphi(a)(y)\}$ is dense (Proposition 3.8(c)). So φ is α-complete (Proposition 2.1(b)). □

Lemma 3.11. *$\varphi: L \to M$ is epic in $\mathscr{W}(\alpha)$ if and only if $Y(\varphi): Y(M) \to Y(L)$ is monic in α-SpFi.*

Proof. Let φ be epic in $\mathscr{W}(\alpha)$ and suppose for $i = 1, 2$ that $f_i : X \to Y(M)$ are α-SpFi morphisms, and $Y(\varphi) \circ f_1 = Y(\varphi) \circ f_2$.

We utilize a result of Gleason:

For each X, there is an essentially unique pair (EX, π) such that $\pi: EX \twoheadrightarrow X$ is ∞-irreducible (Definition 3.14) and EX is extremally disconnected [12]. (EX, π) is called the *absolute* of X [25]. The map π is also an ∞-SpFi morphism (Lemma 3.15). Since EX is extremally disconnected, $D(EX)$ is a $\mathscr{W}$ object [17]. For $i = 1, 2$, let $h_i = f_i \circ \pi$. The h_i are α-SpFi morphisms. Define $\gamma_i : M \to D(EX)$, for $i = 1, 2$, by $\gamma_i(b) = b \circ h_i$ for $b \in M$. This definition makes sense because $M \subseteq D(Y(M))$ and so $b^{-1}(\mathbb{R}) \in \mathscr{G}_{\omega_1}(Y(M))$ (i.e., $\gamma_i(b) \in D(X)$ because $\gamma_i(b)^{-1}(\mathbb{R}) = h_i^{-1}(b^{-1}(\mathbb{R}))$ is dense in EX by virtue of the map h_i being an α-SpFi morphism). Clearly, γ_i is a homomorphism, and by Theorem 3.3(a), $Y(\gamma_i) = h_i$. Therefore the γ_i are α-complete (Lemma 3.10). Moreover, $Y(\gamma_1 \circ \varphi) = Y(\varphi) \circ Y(\gamma_1) = Y(\varphi) \circ Y(\gamma_2) = Y(\gamma_2 \circ \varphi)$. Therefore because Y is faithful (Theorem 3.3(b)), $\gamma_1 \circ \varphi = \gamma_2 \circ \varphi$. Hence $\gamma_1 = \gamma_2$, and $h_1 = Y(\gamma_1) = Y(\gamma_2) = h_2$. $f_1 = f_2$ because π is onto, so $Y(\varphi)$ is monic in α-SpFi.

Now suppose $Y(\varphi)$ is monic in α-SpFi and suppose for $i = 1, 2$, that $\gamma_i : M \to N$ are α-complete with $\gamma_1 \circ \varphi = \gamma_2 \circ \varphi$. Then $Y(\varphi) \circ Y(\gamma_1) = Y(\varphi) \circ Y(\gamma_2)$. Since $Y(\gamma_i)$ is an α-SpFi morphism (Lemma 3.10), we have by our assumption that $Y(\gamma_1) = Y(\gamma_2)$. And so $\gamma_1 = \gamma_2$. Therefore φ is epic. □

Theorem 3.12 [5]. *The following are equivalent:*

(a) *X is α-quasi-F (Definition 1.5).*

(b) *Each $G \in \mathscr{G}_\alpha(X)_\delta$ is C^*-embedded in X.*

(c) *$C(X)$ is α-Dedekind complete.*

Theorem 3.13. *If L is α-Dedekind complete, then $Y(L)$ is α-quasi-F.*

Proof. Let $K \subset Y(L)$ be a dense α-Lindelöf subspace and let $h \in C^*(K)$. Without loss of generality, assume $0 \leq h \leq 1$. For each $n \in \mathbb{N}$ and $p \in K$ there exists a neighborhood, V_p^n, of p in $Y(L)$ such that $\max h[\bar{V}_p^n \cap K] - \min h[\bar{V}_p^n \cap K] < n^{-1}$. Choose a neighborhood, U_p^n, of p in $Y(L)$ so that $\bar{U}_p^n \subset V_p^n \subset \bar{V}_p^n$. Let $M = \max h[\bar{V}_p^n \cap K]$

and $m = \min h[\bar{V}_p^n \cap K]$. Since $L^* \subseteq C(Y(L))$ separates the points of $Y(L)$ there are functions $a_p^n, b_p^n \in L^*$ such that:

(i) $0 \leq a_p \leq m$; $M \leq b_p \leq 1$.

(ii) $a_p^n[\bar{U}_p^n] = m$; $a_p^n[Y(L) \setminus V_p^n] = 0$.

(iii) $b_p^n[\bar{U}_p^n] = M$; $b_p^n[Y(L) \setminus V_p^n] = 1$.

It follows that $a_{p'}^{n'}(y) < h(y) < b_p^n(y)$ for all $n, n' \in \mathbb{N}$ and $p, p', y \in K$.

Because K is α-Lindelöf we have, for each $n \in \mathbb{N}$, a subset $P' \subset K$ such that $|P'| < \alpha$ and $K = \bigcup_{P'} \{U_{p'}^n\}$. Let $A = \{a_{p'}^n : n \in \mathbb{N}, p' \in P'\}$ and $B = \{b_{p'}^n : n \in \mathbb{N}, p' \in P'\}$. It follows that $|A|, |B| < \alpha$ and $B \geq A$. We claim $0 = \bigwedge^L (B - A)$. For $K \subset \{x \in X : 0 = \bigwedge_{A,B}^{\mathbb{R}} (b(x) - a(x))\}$, and since K is dense in $Y(L)$ we have, as in Lemma 3.9(a), that $0 = \bigwedge^L (B - A)$. Therefore, because L is α-Dedekind complete, there is a $g \in L^* \subseteq C(Y(L))$ with $g = \bigwedge B = \bigvee A$. Clearly, $g | K = h$, hence K is C^*-embedded in $Y(L)$. □

Definition 3.14 below shows how Y links the notions of α-density and α-irreducibility. Lemma 3.15 is a slight generalization of the equivalence of 3.12(a) and 3.12(g) in [5].

Definition 3.14. Let $f : X \twoheadrightarrow Y$. f is said to be *α-irreducible* if for each $U \in \mathrm{Coz}(X)$, there is a $V \in \mathrm{Coz}_\alpha(Y)$ such that $f^{-1}(V) \subset U$ and $\overline{f^{-1}(V)} = \bar{U}$.

Lemma 3.15. *An embedding $\varphi : L \hookrightarrow M$ is α-dense if and only if $Y(\varphi) : Y(M) \twoheadrightarrow Y(L)$ is α-irreducible.*

Proof. Suppose $Y(\varphi)$ is α-irreducible, $b \in M^+$, and $U = \mathrm{coz}(b) \cap b^{-1}(\mathbb{R})$. Since U is ω_1-Lindelöf ($b^{-1}(\mathbb{R}) \in \mathrm{Coz}(Y(M))$) and b does not take the value $\pm\infty$ anywhere on U, there is for each $n \in \mathbb{N}$ a family $\{U_k^n : k \in \mathbb{N}\} \subset \mathrm{Coz}(Y(M))$ such that $U = \bigcup_k U_k^n = \bigcup_k \bar{U}_k^n$, and $\max b[\bar{U}_k^n] - \min b[\bar{U}_k^n] \leq n^{-1}$ for each $k \in \mathbb{N}$.

The indices will now get a bit hairy.

Since $Y(\varphi)$ is α-irreducible, we have for each $(n, k) \in \mathbb{N} \times \mathbb{N}$, a $V_k^n \in \mathrm{Coz}_\alpha(Y(L))$ such that $Y(\varphi)^{-1}(V_k^n)$ is dense in U_k^n. Also, since for each (n, k), V_k^n is α-Lindelöf, there is a family, $\{V_k^n(i) : i \in I, |I| < \alpha\} \subset \mathrm{Coz}(Y(L))$, such that $V_k^n = \bigcup_i \bar{V}_k^n(i)$. Finally, there is a family of functions in L, $\{a_k^n(i) : (i, n, k) \in I \times \mathbb{N} \times \mathbb{N}\}$, such that for each (i, n, k):

(i) $0 \leq a_k^n(i) \leq \min b[\bar{U}_k^n]$.

(ii) $a_k^n(i)[\bar{V}_k^n(i)] = \min b[\bar{U}_k^n]$.

(iii) $a_k^n(i)[Y(L) \setminus V_k^n] = 0$.

Upon careful reflection (and recalling that $a_k^n(i) \circ Y(\varphi)(y) = \varphi(a_k^n(i))(y)$) one will see that

$$\bigcap_n \left(\bigcup_k \{ Y(\varphi)^{-1}(V_k^n) \cup (Y(M) \setminus U)^\circ \} \right)$$
$$\subset \left\{ y \in Y(M) : \bigvee_{i,n,k} a_k^n(i) \circ Y(\varphi)(y) = b(y) \right\}.$$

Invoking the Baire category theorem, one can see that the former, and hence, the latter set is dense in $Y(M)$ because for each $n \in N$, $\bigcup_k Y(\varphi)^{-1}(V_k^n)$ is dense in U; so $\bigcup_k \{Y(\varphi)^{-1}(V_k^n) \cup (Y(M) \setminus U)^\circ\}$ is dense and open in $Y(M)$. Therefore

$$\left\{ y \in Y(M) \colon \bigvee_{i,n,k} a_k^n(i) \circ Y(\varphi)(y) = b(y) \right\}$$

$$= \left\{ y \in Y(M) \colon \bigvee_{i,n,k} \varphi(a_k^n(i))(y) = b(y) \right\}$$

is dense in $Y(M)$. Applying Lemma 3.9(a), we have that $b = \bigvee_{i,n,k}^L \varphi(a_k^n(i))$. Since $|I \times \mathbb{N} \times \mathbb{N}| < \alpha$, it follows that φ is α-dense.

On the other hand, suppose φ is an α-dense embedding. Let $U \in \mathrm{Coz}(Y(M))$. We can assume without loss of generality that $U = \mathrm{coz}(b)$ for some $b \in M^+$. Since φ is α-dense, there is a $A \subset L$ with $|A| < \alpha$ such that $b = \bigvee^M \varphi[A]$. If we let $P_b = \{y \in Y(M) \colon \bigvee_A^M \varphi(a)(y) = b(y)\}$, then P_b is dense in $Y(M)$ (Lemma 3.9(a)). We claim that for $V = (\bigcup_A \mathrm{coz}(a)) \in \mathrm{Coz}_\alpha(Y(L))$, we have that $Y(\varphi)^{-1}(V)$ is dense in U. Now

$$\mathrm{coz}(b) \cap P_b \subset \bigcup_A \mathrm{coz}(\varphi(a))$$

$$= \bigcup_A \mathrm{coz}(a \circ Y(\varphi)) = Y(\varphi)^{-1}(V) \subset \mathrm{coz}(b) = U.$$

Since P_b is dense in $Y(M)$, $Y(\varphi)^{-1}(V)$ is dense in U. □

The following is a generalization to arbitrary α of a lemma of Weinberg [29].

Corollary 3.16. *$f: X \twoheadrightarrow Y$ is α-irreducible if and only if $f': C(Y) \hookrightarrow C(X)$ is α-dense. Moreover, if f is α-irreducible, then f' is an α-*SpFi *morphism.*

Proof. $Y(f') = f$. Apply Definition 3.14, Corollary 2.7, and Lemma 3.10. □

Note, Lemma 3.15 follows directly from [5, 3.12].

4. The α-quasi-F cover

In this section all spaces are assumed to be compact and Hausdorff. Theorem 4.1 below is a synthesis of results taken from [5, 9, 18, 24]. See also [13, 22, 23, 27]. Recall the definition of the minimum P cover of X (Definition 1.6). Here we take $P = \alpha$-quasi-F.

Theorem 4.1. *For each X, there exists a minimum α-quasi-F cover, denoted by $(\mathrm{QF}_\alpha X, q_\alpha)$. Moreover, q_α is α-irreducible; and any α-quasi-F cover (Y, f) of X with f α-irreducible is essentially unique.*

Theorem 4.2. *Let $(i, \widehat{C(X)_\alpha})$ be the α-Dedekind complete epireflection of $C(X)$ in $\mathscr{W}(\alpha)$ (Theorem 2.19). Then $(Y(\widehat{C(X)_\alpha}), Y(i))$ is essentially identical to the α-quasi-F cover of X.*

Proof. Apply Theorem 3.13, Definition 3.14, and Theorem 4.1. □

We now give the main result of this section. We say (Y, f) is an *α-irreducible preimage* of X if $f: X \to Y$ is α-irreducible. Recall that in an abstract category $\mathscr{B}$, a full subcategory $\mathscr{A}$ is called *monocoreflective* if for each $B \in |\mathscr{B}|$ there is an $A_B \in |\mathscr{A}|$ and a monic $m_B: A_B \to B$ such that whenever $f: A \to B$ and $A \in |\mathscr{A}|$ there is an $\bar{f}: A \to A_B$ with $f = m \circ \bar{f}$.

Theorem 4.3. (a) *Suppose (Y, f) is an α-irreducible preimage of X; $h: Z \to X$ is* α-SpFi; *and Z is α-quasi-F. Then there exists a unique* α-SpFi *morphism $\bar{h}: Z \to Y$ with $h = f \circ \bar{h}$.*

(b) *In* α-SpFi, *the full subcategory of α-quasi-F spaces is monocoreflective, and* $(\mathrm{QF}_\alpha X, q_\alpha)$ *is the α-quasi-F monocoreflection of X.*

(c) $(\mathrm{QF}_\alpha X, q_\alpha)$ *is the maximal α-irreducible preimage of X. That is, if (Y, f) is an α-irreducible preimage of X, then there is an $\hat{f}: \mathrm{QF}_\alpha X \twoheadrightarrow Y$ such that $q_\alpha = f \circ \hat{f}$. It then follows that $\hat{f}$ must be α-irreducible and* $\mathrm{QF}_\alpha X \cong \mathrm{QF}_\alpha Y$.

Proof. (a) Let (Y, f) be as above. By Lemma 3.15, $(f', C(Y))$ is an α-dense extension of $C(X)$. By Lemma 3.10 and Theorem 3.12 respectively, if $h: Z \to X$ is an α-SpFi morphism and Z is α-quasi-F, then $h': C(X) \to C(Z)$ is α-complete and $C(Z)$ is α-Dedekind complete. Hence by Theorem 2.16, there is a unique α-complete $\varphi: C(Y) \to C(Z)$ such that $f' = \varphi \circ h'$. Applying the Yosida functor we get that $Y(\varphi): Z = Y(C(Z)) \to Y(C(Y)) = Y$ is an α-SpFi morphism and $h = Y(h') = Y(f') \circ Y(\varphi) = f \circ Y(\varphi)$. Take $\bar{h} = Y(\varphi)$.

(b) It follows from Proposition 2.8, Lemma 3.11 and Definition 3.14 that q_α is monic in α-SpFi. Apply (a) here and Theorem 4.1.

(c) By (a) here we have a map $\hat{f}: \mathrm{QF}_\alpha X \to Y$ such that $q_\alpha = \hat{f} \circ f$. It is not hard to see that since q_α and f are α-irreducible, then $\hat{f}$ must also be α-irreducible. Finally, to show $\mathrm{QF}_\alpha X \cong \mathrm{QF}_\alpha Y$, apply Theorem 4.1. □

Theorem 4.3(b) was proved independently by A. Molitor in [23], and Theorem 4.3(c) has been demonstrated in [18] when $\alpha = \omega_1$.

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